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Bosonization of non-relativstic fermions in 2-dimensions and collective field theory

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Abstract: We revisit bosonization of non-relativistic fermions in one space dimension. Our motivation is the recent work on bubbling half-BPS geometries by Lin, Lunin and Maldacena [13]. After reviewing earlier work on exact bosonization in terms of a non-commutative theory, we derive an action for the collective field which lives on the droplet boundaries in the classical limit. Our action is manifestly invariant under time-dependent reparametrizations of the boundary. We show that, in an appropriate gauge, the classical collective field equations imply that each point on the boundary satisfies Hamilton's equations for a classical particle in the appropriate potential. For the harmonic oscillator potential, a straightforward quantization of this action can be carried out *exactly* for any boundary profile. For a finite number of fermions, the quantum collective field theory does not reproduce the results of the exact noncommutative bosonization, while the latter are in complete agreement with the results computed directly in the fermi theory.

Keywords: Field Theories in Lower Dimensions, AdS-CFT and dS-CFT Correspondence, Non-Commutative Geometry.

JHEP07(2005)064

Contents

1. Introduction

The connection between free non-relativistic fermions and string theory in 2-dimensions is known since early nineties [1−9].[1] Recent studies [13−26] have shown that free non-relativistic fermions also appear in other situations in string theory, typically in sectors which have high enough supersymmetry. The requirement of sufficient amount of supersymmetry is understandable since 2-dimensional free fermions form an integrable system. The correspondence with fermions is usually more easy to see in the dual field theory. Remarkably, in [13] the authors found a class of 1/2−BPS solutions of supergravity equations in one-to-one correspondence with classical configurations of free non-relativistic fermions in 2-dimensions. While providing yet another example of AdS/CFT correspondence,[2] it opens up the interesting possibility of learning something about the nature of quantum gravity and string theory from free fermions [28, 23]. Since in [13] fermions make contact

[1] For an older review of the subject see [10]; more recent review and references can be found in [11, 12].
[2] For a review see [27].

with geometry via the bosonized theory which describes their collective motion, it is essential to understand all aspects of the bosonized theory in order to be able to use the full power of free fermions. This provides the main motivation for the present work.

The organization of this paper is as follows. In section 2 we review (see also [32]) the works [29 − 31] in which an exact bosonization of free non-relativistic fermions in 2-dimensions has been developed in terms of a noncommutative theory using Wigner phase space density. In section 3 we discuss the classical limit of this bosonized theory. In the classical limit, a generic configuration consists of droplets of fermi fluid on the phase plane. The dynamics, which is associated to the collective motion of the droplets, manifests itself in their changing boundaries. The action for this collective motion has a built in symmetry under arbitrary time-dependent reparametrizations of the droplet boundaries. This gauge symmetry reflects the fact that the motion of the fluid along the boundaries of the droplets is unphysical because of the indistinguishability of fermions. In this section, we derive a manifestly gauge-invariant classical action which describes the boundary dynamics of the droplets for arbitrary boundary profiles. Issues related to gauge fixing for different droplet boundary profiles are also discussed in this section. In an appropriate gauge, we show that classically each point on the boundary simply follows Hamilton's equations of motion. Quantization of the collective field theory is carried out in section 4. For the harmonic oscillator potential, quantization can be carried out exactly for any droplet boundary profile. We find that for a finite number of fermions, the spectrum of the collective quantized theory does not agree with the exact spectrum at large energies. Furthermore, the phase space density fails to reproduce the precise details of the exact result. We summarize our results and end with some concluding remarks in section 5.

2. Review of exact bosonization

Consider N free non-relativistic fermions moving in one space dimension in a potential. The fermion wavefunctions satisfy the Schroedinger equation

$$i\hbar\,\partial_t\psi(x,t) = H\psi(x,t)\,, \tag{2.1}$$

where the single-particle hamiltonian H is given by

$$H = \frac{1}{2}(-\hbar^2\partial_x^2 + V(x))\,. \tag{2.2}$$

Moreover, the fixed fermion number constraint is,

$$N = \int_{-\infty}^{+\infty} dx \sum_{m=1}^{N} \psi_m^*(x,t)\psi_m(x,t)\,. \tag{2.3}$$

Here $\psi_m(x,t)$ $(m = 1, 2, 3, \ldots)$ form a complete orthonormal set of single-particle wavefunctions, i.e. $\delta_{mn} = \int_{-\infty}^{+\infty} dx\,\psi_m^*(x,t)\psi_n(x,t)$.

2.1 Bosonization in terms of Wigner density

The bosonization carried out in [29 − 31] uses the Wigner phase space density as its basic building block. In terms of the fermion wavefunctions, it is defined by the expression

$$u(p,q,t) = \int_{-\infty}^{+\infty} dx\, e^{-ipx/\hbar} \sum_{m=1}^{N} \psi_m^*\left(q-\frac{x}{2},t\right)\psi_m\left(q+\frac{x}{2},t\right). \tag{2.4}$$

A fundamental property of the Wigner density, one that captures the underlying fermionic nature of the degrees of freedom in the bosonized version, is that it satisfies a nonlinear constraint. This constraint, which can be elegantly written using the non-commutative star product[3] in the phase plane, is

$$u * u(p,q,t) = u(p,q,t)\,, \tag{2.5}$$

where the star product is defined in the usual way,

$$u_1 * u_2(p,q,t) = \left[e^{\frac{i\hbar}{2}(\partial_{q_1}\partial_{p_2}-\partial_{q_2}\partial_{p_1})}u_1(p_1,q_1,t)u_2(p_2,q_2,t)\right]_{q_1=q_2=q,\,p_1=p_2=p}. \tag{2.6}$$

A quick way of deriving the constraint is to first construct the bilocal fermion bilinear $\sum_{m=1}^{N}\psi_m^*(x,t)\psi_m(y,t) \equiv \phi(x,y,t)$. As a consequence of the orthonormality of the fermion wavefunctions, this bilocal function satisfies the constraint $\int_{-\infty}^{+\infty} dz\, \phi(x,z,t)\phi(z,y,t) = \phi(x,y,t)$. The definition of u, eqn(2.4), which can be easily rewritten in terms of the bilocal function $\phi(x,y,t)$, and some simple algebraic manipulations then lead to the constraint (2.5) on u.

In addition to the above constraint, u satisfies the condition

$$N = \int \frac{dpdq}{2\pi\hbar}\, u(p,q,t)\,, \tag{2.7}$$

which is just a restatement, in terms of the Wigner density u, of the fact that the total number of fermions is N. Finally, an equation of motion can be derived for u using the Schroedinger equation satisfied by the fermion wavefunctions. One gets,

$$\partial_t u(p,q,t) = \{h,u\}_*(p,q,t)\,. \tag{2.8}$$

Here $h = \frac{1}{2}(p^2 + V(q))$ is the classical single-particle hamiltonian. The bracket on the right-hand side is Moyal's generalization of the Poisson bracket involving the star product, namely $\{h,u\}_* = \frac{1}{i\hbar}(h*u - u*h)$. In the limit $\hbar \to 0$ the Moyal bracket goes over to the Poisson bracket.

2.2 The role of $\mathbf{W}_\infty$ symmetry

Notice that the constraints (2.5) and (2.7) are left unchanged by the following infinitesimal variation of u,

$$\delta u = \{\epsilon, u\}_*\,, \tag{2.9}$$

[3]The compact star product notation was not used in references [29 − 31]. The expressions given there are however the same with star products written out in long-hand.

JHEP07(2005)064

where $\epsilon = \epsilon(p,q,t)$. This is, in fact, the most general variation that leaves these two constraints unchanged. If the t-dependence of ϵ is such that $\partial_t \epsilon = \{h, \epsilon\}_*$, then $u + \delta u$ also satisfies the equation of motion (2.8). It is easy to see that two such successive transformations satisfy the group composition law. In fact, the relevant group is W_∞, the quantum generalization of the group of area-preserving diffeomorphisms in 2-dimensions, and these transformations move us on a co-adjoint orbit of W_∞ in the configuration space of u's. In particular, changes in u corresponding to t-independent ϵ's satisfying $\{h, \epsilon\}_* = 0$ are symmetries. The corresponding conserved charges[4] are

$$Q_n = \int \frac{dpdq}{2\pi\hbar} \qquad h^n * u(p,q,t)\,, \quad n = 0, 1, 2, \dots\,, \tag{2.10}$$

where $h^n = h * h * \cdots * h$ has n factors.

The charges Q_0 and Q_1 are familiar. They measure the total number of fermions and total energy, respectively. Other charges are less familiar in the bosonic version. In terms of fermions, however, they can be easily seen to be sums of higher (than linear) powers of individual fermion energies, which are obviously conserved in the non-interacting theory that we are considering here.[5]

2.3 The action of the bosonized theory

Equations (2.4)–(2.8) constitute an exact bosonization of the fermion problem. In [29 − 31] a derivation of this bosonization has been given and an action obtained for the Wigner density, whose variation gives rise to the equation of motion (2.8). The derivation uses coherent states and the coadjoint orbits of W_∞, much like the coherent states and coadjoint orbits of SU(2) are used in the case of a spin in a magnetic field. As in the latter case, one needs to construct a "cap" action, the cap being parametrized by time t and an additional variable s ($0 \le s \le 1$), such that $u(p,q,t) = u(p,q,t; s = 1)$ while $u(p,q,t; s = 0) = \bar{u}(p,q)$ is a t-independent function. One gets the following action:

$$S = \int dt \int \frac{dpdq}{2\pi\hbar} \left(\int_0^1 ds\, \hbar^2 u * \{\partial_t u, \partial_s u\}_* - u * h \right). \tag{2.11}$$

It can be easily verified that the equation of motion (2.8) follows from this action upon using the variation (2.9) and the constraints (2.5) and (2.7).

A simpler form for the action can be obtained, one that is useful for going over to the classical limit, if one rewrites the action in terms of a "reference" density u_0 (for example,

[4]Similar charges first appeared in [5] in a system of free fermions in inverted harmonic oscillator potential, which is relevant for the $c = 1$ matrix model.

[5]In the fermionic theory, all the charges in any given state can be explicitly written in terms of the individual energies of N fermions. One might wonder how the bosonic version of these charges in (2.10) depends on only N independent parameters. The point is that the u's that should be used to evaluate these charges must satisfy the constraints (2.5) and (2.7). Once this is ensured, it can be seen that the charges Q_n can be expressed in terms of only N independent parameters.

it could be the density in the fermi vacuum) and the W_∞ group element v[6] that is needed to "rotate" u_0 to $u = v * u_0 * v^\dagger$. One gets

$$S = \int dt \int \frac{dpdq}{2\pi\hbar} \qquad u * (a_t - h)\,, \tag{2.12}$$

where $a_t \equiv i\hbar\,\partial_t v * v^\dagger$. Since a_t also satisfies the equation

$$\partial_t u = \{a_t, u\}_*\,, \tag{2.13}$$

which does not depend on the reference density u_0, we may alternatively use this equation to define a_t. We will later use the equations (2.12) and (2.13) to take the classical limit of this bosonized theory.

2.4 Consistency of bosonization

We will end this section by arguing that the equations (2.5)–(2.8) provide an exact bosonization in the sense that solutions to these equations are in one-to-one correspondence with the states of the fermion system. Consider first the constraint equation (2.5). General solutions to this equation have been obtained in [33]. For the harmonic oscillator potential for which $h = \frac{1}{2}(p^2+q^2) \equiv \rho$, which is the case of interest in [13], the analysis is particularly simple. This is because in this case t-independent solutions of (2.8), which correspond to energy eigenstates, are radial functions in the phase plane. As discussed in [33], there are an infinite number of real independent radial solutions to the equation (2.5). The general form of the solution is

$$u = \sum_{m=0}^{\infty} c_m \phi_m(\rho)\,,$$

where $\phi_m(\rho) = 2(-1)^m e^{-2h} L_m(4\rho)$, L_m being the mth Laguerre polynomial. The coefficients c_m can take the values $0, 1$. The fermion number constraint (2.7) fixes the number of nonzero c_m's to be precisely N. These, in fact, identify the filled levels, so it is clear that solutions to the equations (2.5)–(2.8) are in one-to-one correspondence with the energy eigenstates of the fermion theory. In fact, given a time-independent $u(p, q)$ which solves the equations (2.5)–(2.8), one can uniquely reconstruct the filled levels. Since u is essentially a projection operator, finding the corresponding fermi state amounts to finding the subspace on which it projects:

$$\int \frac{dpdq}{2\pi\hbar}\, e^{ip(x-q)/\hbar}\, u\Big(p, \frac{x+q}{2}\Big)\psi(q) = \psi(x)\,. \tag{2.14}$$

For N fermions, this subspace is N-dimensional, so equation (2.14) should have N linearly independent solutions. In this way one can obtain the wavefunctions for the filled states from the given u.

[6] v is unitary, that is $v * v^\dagger = v^\dagger * v = I$.

JHEP07(2005)064

We give below the solution for the Wigner density in the fermi vacuum:

$$u_F = \sum_{m=0}^{N-1} 2(-1)^m e^{-2\rho/\hbar} L_m(4\rho) \,. \tag{2.15}$$

The expression on the right hand side can be rewritten[7] as an integral over a single Laguerre polynomial:

$$u_F = \int_{4\rho/\hbar}^{\infty} dy\, e^{-y/2} L^1_{N-1}(y) \,. \tag{2.16}$$

Note that for large-N [34],

$$e^{-y/2} L^1_{N-1}(y) = \pi^{-1/2} y^{-3/4} N^{1/4} \cos\left(2\sqrt{(N-1)y} - \frac{3\pi}{4}\right) + \mathrm{O}(N^{-1/4}) \,. \tag{2.17}$$

We will make use of these expressions when we compare u_F given above with the one obtained by quantizing the classical limit of the action in (2.13).

3. Classical limit and collective action

The easiest way to take the classical limit of the bosonized theory obtained as outlined above is through the equations (2.12) and (2.13), supplemented by the constraints (2.5) and (2.7). To be precise, in the classical limit we will set $\hbar \to 0$ and $N \to \infty$ while keeping $\hbar N = \rho_0$ fixed. In this limit equation (2.5) becomes $u^2 = u$, whose standard solutions are filled and empty regions of phase plane. Equation (2.7) fixes the total area of the filled regions, $\int dpdq\, u = 2\pi\rho_0$. Note also that the classical energy of any configuration diverges in this limit as $\mathrm{O}(1/\hbar)$.

3.1 Classical motion is area-preserving diffeomorphisms

Let us now first consider equation (2.13). In the classical limit, the Moyal bracket on the right-hand side reduces to the Poisson bracket. Integrating this equation over an infinitesimal amount of time δt gives

$$u(p - \delta t\, \partial_q a_t, q + \delta t\, \partial_p a_t, t + \delta t) \approx u(p, q, t) \,.$$

This equation shows that the classical motion of the fermi fluid is determined by area preserving diffeomorphisms in phase space. Here the diffeomorphism is generated by the function a_t. Physical motion of the fermi fluid is manifested only in changing boundaries[8] of the filled regions. Since the point (p, q) in the phase plane moves to the point $(p - \delta t\, \partial_q a_t, q + \delta t\, \partial_p a_t)$ in an infinitesimal time δt, for consistency we must have

$$\partial_t p = -\partial_q a_t \,, \qquad \partial_t q = \partial_p a_t \,. \tag{3.1}$$

We will use these equations below to get an explicit form for the classical action involving only the boundary variables.

[7] Actually the answer depends on whether N is even or odd. The density falls off to zero at infinity only for N odd. It is this answer that we have given in the equation below.

[8] This is because in the interior of the filled regions, the motion is unphysical since the fermions filling the interior are identical. Even on the boundary of a filled region, motion of the fluid along the boundary is unphysical for the same reason.

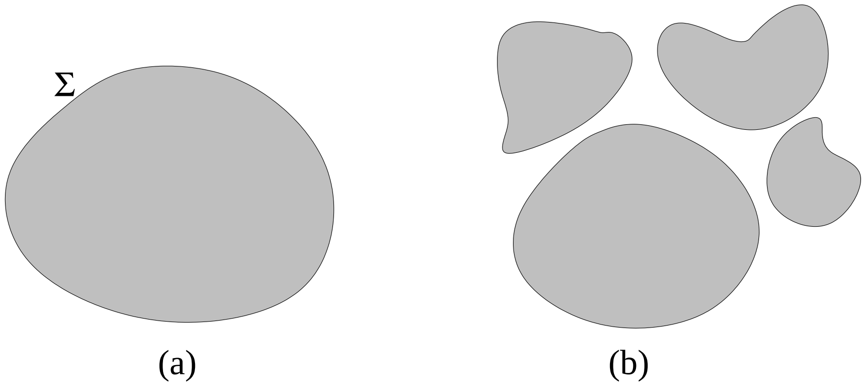

Figure 1: (a) A simple configuration with a single connected filled region Σ in phase space. The droplet is centered at the origin in the phase plane. (b) A general configuration of several disconnected fluid droplets on the phase plane.

3.2 The boundary action

Consider the action in (2.12) in the classical limit. For simplicity we will assume that the density u is nonzero only in a single connected region Σ, centered at the origin of the phase plane, as shown in figure 1a. Generalization to several disconnected filled regions, as for example in figure 1b, is straightforward. Let us parametrize the filled region Σ by (τ, σ) which take values in the unit disc $0 \leq \tau \leq 1, \quad 0 \leq \sigma \leq 2\pi$. Then the classical action can be written as an action on this disc

$$S = \frac{1}{2\pi\hbar} \int dt \int_0^1 d\tau \int_0^{2\pi} d\sigma \qquad (\partial_\sigma p \partial_\tau q - \partial_\sigma q \partial_\tau p)(a_t - h)\,. \tag{3.2}$$

For later purposes, it will be more convenient for us to work in polar coordinates,[9] $q = r\cos\theta$, $p = r\sin\theta$. In these coordinates the above action becomes

$$S = \frac{1}{2\pi\hbar} \int dt \int_0^1 d\tau \int_0^{2\pi} d\sigma \qquad (\partial_\sigma \theta \partial_\tau \rho - \partial_\sigma \rho \partial_\tau \theta)(a_t - h)\,, \tag{3.3}$$

where we have used $r^2/2 \equiv \rho$. We also write down the consistency conditions (3.1) in these coordinates:

$$\partial_t \theta = -\partial_\rho a_t\,, \qquad \partial_t \rho = \partial_\theta a_t\,. \tag{3.4}$$

Since physical motion of the fermi fluid is manifested only in changing boundary of the filled region Σ, it should be possible to reexpress this classical action in terms of appropriate degrees of freedom which live only on the boundary of the disc. To do so, let us introduce the collective field[10] ϕ, which is defined by the relation

$$\rho \partial_\sigma \theta = \partial_\sigma \phi\,. \tag{3.5}$$

[9]For applications to 2-dimensional string theory and the $c = 1$ matrix model, it would be more appropriate to work in hyperbolic coordinates.

[10]The collective field approach to approximate bosonization of non-relativistic fermions was first used in [35].

In terms of this variable,

$$(\partial_\sigma\theta\partial_\tau\rho - \partial_\sigma\rho\partial_\tau\theta) = \partial_\sigma(\partial_\tau\phi - \rho\partial_\tau\theta)\,.$$

Also, using the consistency conditions (3.4), we get

$$\begin{aligned}\partial_\sigma a_t &= \partial_t\rho\partial_\sigma\theta - \partial_t\theta\partial_\sigma\rho \\ &= \partial_\sigma(\partial_t\phi - \rho\partial_t\theta)\,.\end{aligned}$$

With the help of these two relations, and after some algebraic manipulations involving partial integrations, the integrands of both the kinetic term involving a_t as well as that of the hamiltonian piece in action (3.3) can be expressed as total derivatives in τ. The resulting boundary action (at $\tau = 1$) is

$$S = \frac{1}{2\pi\hbar}\int dt \int_0^{2\pi} d\sigma \left[\frac{1}{2}\partial_\sigma\phi(\partial_t\phi - \rho\partial_t\theta) - \partial_\sigma\theta\tilde{h}\right], \tag{3.6}$$

where $\partial_\rho\tilde{h} = h$.[11] This action, together with the fixed area constraint, $\int d\sigma\ \partial_\sigma\theta\rho = 2\pi\rho_0$, describes the dynamics of the boundary of the filled region Σ.

3.3 Boundary reparametrizations

Remarkably, the equations of motion arising from the above action for independent variations with respect to ϕ and θ turn out to be *identical*. This equation is

$$\partial_\sigma\theta\partial_t\rho - \partial_t\theta\partial_\sigma\rho - \partial_\sigma h = 0\,. \tag{3.7}$$

It is not hard to understand the reason for this. The action (3.6) is invariant under t-dependent σ-reparametizations, $\sigma \to \sigma'(t,\sigma)$. This gauge invariance of S arises because the motion of the fluid along the boundary is unphysical, which is due to the indistinguishability of the underlying fermionic degrees of freedom. As a consequence of this gauge symmetry, $\rho(t,\sigma)$ and $\theta(t,\sigma)$ provide a redundant description of the dynamics of the boundary. The physical description of the dynamics requires only "half" the number of variables. The single equation of motion (3.7) describes the dynamics of this physical, gauge-invariant degree of freedom.

For a generic potential (3.7) is a complicated non-linear partial differential equation, not easy to solve. However, for the harmonic oscillator potential it simplifies dramatically. In this case $h = \rho$, so the equation becomes

$$\partial_\sigma\theta\partial_t\rho - (1 + \partial_t\theta)\partial_\sigma\rho = 0\,. \tag{3.8}$$

A general solution to this equation can be written down immediately. At points on the boundary where neither $\partial_\sigma\rho$ nor $\partial_\sigma\theta$ vanishes, the solution is

$$\rho(t,\sigma) = f(t + \theta(t,\sigma))\,. \tag{3.9}$$

[11] For example, for the harmonic oscillator potential, $h = \frac{1}{2}(p^2 + q^2) = \rho$ and so $\tilde{h} = \rho^2/2$, upto an irrelevant constant.

JHEP07(2005)064

Here f is any arbitrary periodic function (in $\theta \to \theta + 2\pi$), subject only to the fixed area constraint $\int d\sigma\ \partial_\sigma \theta \rho = 2\pi\rho_0$.

The above solution is not valid at the points σ_i where either $\partial_\sigma \rho$ or $\partial_\sigma \theta$ vanishes. At these points (3.9) requires *both* $\partial_\sigma \rho$ and $\partial_\sigma \theta$ to vanish together, which is not possible.[12] The equation of motion is, however, satisfied everywhere. Therefore, at the points where $\partial_\sigma \rho$ vanishes, we must have $\partial_t \rho = 0$. Similarly, at the points where $\partial_\sigma \theta$ vanishes, we must have $\partial_t \theta = -1$. To proceed further and completely solve the classical dynamics of the equation (3.8), we need to suitably gauge-fix its symmetry under t-dependent σ-reparametizations. This is what we will do now.

3.4 Gauge-fixing boundary reparametrizations

We will continue our discussion with the specific example of the harmonic oscillator potential. Generalization to other potentials is easy and will be mentioned at the end. We start by choosing a gauge in which ρ is a fixed t-independent function of σ. This choice is possible, except at points where $\partial_\sigma \rho$ vanishes, since ρ is gauge-invariant at these points. However, by equation of motion (3.8), ρ is still t-independent at these points. Thus, in this gauge it is consistent with the equation of motion to choose $\rho(t,\sigma) = \bar{\rho}(\sigma)$ everywhere. From (3.8) it follows that we must have $\theta(t,\sigma) = -t + \bar{\theta}(\sigma)$, except possibly at the points where $\partial_\sigma \rho$ vanishes. But by a gauge choice we can adjust θ to this solution at these points as well.[13] Hence, in this gauge, the complete gauge-fixed solution of (3.8) is[14]

$$\rho(t,\sigma) = \bar{\rho}(\sigma)\,, \qquad \theta(t,\sigma) = -t + \bar{\theta}(\sigma)\,. \tag{3.10}$$

We see that each point on the boundary of the filled region simply rotates in a circle around the origin in the phase plane. This is precisely what happens in the fermi picture in the classical limit - in the harmonic oscillator potential the particles simply rotate in circles whose radii are determined by their energies.

It is easy to generalize the above argument to other potentials. Let us rewrite the equation of motion (3.7) for a generic potential as follows:

$$\partial_\sigma \theta(\partial_t \rho - \partial_\theta h) - \partial_\sigma \rho(\partial_t \theta + \partial_\rho h) = 0\,. \tag{3.11}$$

As above, we gauge-fix ρ such that its t-dependence is determined by the equation $\partial_t \rho - \partial_\theta h = 0$. By (3.11) this equation continues to be satisfied even at points where $\partial_\sigma \rho$ vanishes and ρ is gauge-invariant. Then, from (3.11) we get $\partial_t \theta + \partial_\rho h = 0$ and by gauge-fixing θ we can enforce this equation even at points where $\partial_\sigma \rho$ vanishes. The result is that in this gauge, physical dynamics of the boundary is obtained by solving the equations

$$\partial_t \rho = \partial_\theta h\,, \qquad \partial_t \theta = -\partial_\rho h\,. \tag{3.12}$$

[12] There is a possibility of both $\partial_\sigma \rho$ and $\partial_\sigma \theta$ vanishing for self-intersecting curves at the point of self-intersection. However, such boundary profiles do not seem meaningful for regions filled with fermi fluid. Moreover, a self-intersection of the boundary is a pinching of the filled region, which cannot be meaningfully described in the classical limit.

[13] Since ρ is gauge-invariant at the points where $\partial_\sigma \rho$ vanishes, we use the freedom of gauge-transformations to adjust θ at these points.

[14] This solution is consistent with (3.9) for values of σ for which the latter holds.

These are precisely Hamilton's equations for a particle with the hamiltonian h. So the points on the boundary of the filled region follow the trajectories described by the solutions to these equations. This is exactly as expected from fermions moving in the given potential in the classical limit.

We end this discussion by explaining the physical meaning of the above gauge-fixing procedure. At generic points on the boundary of a filled region, the motion of the fermi fluid can be arranged to be purely angular or purely radial by adding an arbitrary motion of the fluid along the boundary. The symmetry under t-dependent σ-reparametizations allows us to do this. At the points σ_i where $\partial_\sigma \rho$ ($\partial_\sigma \theta$) vanishes, however, the tangent to the fluid boundary is purely angular (radial), physical motion is purely radial (angular) and only angular (radial) motion can be changed by gauge changes. We use this latter freedom to adjust, consistent with equation of motion, $\theta(t,\sigma)$ ($\rho(t,\sigma)$) to be a smooth function of t across the points σ_i.

3.5 Conserved charges and area-preserving diffeomorphisms

The conserved charges that we found in (2.10) in the exact bosonized theory have a classical limit. In fact, the classical expressions for the charges can be written entirely in terms of the boundary variables which appear in the action (3.6). We get

$$Q_n^{\mathrm{cl}} = \frac{1}{2\pi\hbar}\int_0^{2\pi} d\sigma\ \partial_\sigma\theta\, \tilde{h}_n\,, \qquad n = 0,1,2,\dots\,, \tag{3.13}$$

where $\partial_\rho \tilde{h}_n = h^n$. One can check directly using the classical equation of motion (3.7) that these charges are conserved. Note that, unlike the charges in (2.10), here the number of independently conserved charges is infinite. This is consistent with the fact that in the classical limit the number of fermions goes to infinity.

In the case of harmonic oscillator potential, these charges take an especially simple form:

$$Q_n^{\mathrm{cl}} = \frac{1}{2\pi\hbar}\int_0^{2\pi} d\sigma\ \partial_\sigma\theta\ \frac{\rho^{n+1}}{(n+1)}\,, \qquad n = 0,1,2,\dots \tag{3.14}$$

In addition, in this case one can give an explicit expression for the other charges of the group of area-preserving diffeomorphisms:

$$W_{nm}^{\mathrm{cl}} = \frac{1}{2\pi\hbar}\int_0^{2\pi} d\sigma\ \partial_\sigma\theta\ \rho^{\frac{n+m}{2}+1}\ e^{i(n-m)\theta}\,, \qquad n,m = 0,1,2,\dots \tag{3.15}$$

Using the equation of motion (3.8) it can be checked that these satisfy the equation

$$\partial_t W_{nm}^{\mathrm{cl}} = i(n-m) W_{nm}^{\mathrm{cl}}\,. \tag{3.16}$$

The "diagonal" charges, W_{nn}^{cl}, are conserved. They are, in fact, essentially the charges Q_n^{cl}.

4. Quantization of the collective theory

In this section we will discuss the problem of quantization of the collective theory. Throughout this section we will limit our discussion to the harmonic oscillator potential. This is

JHEP07(2005)064

because in this case quantization can be done exactly. One can, of course, develop a perturbation expansion in small $\hbar$ for more general potentials, but we will not pursue that here.

Our staring point is the classical action (3.6). Canonical quantization requires that we first fix a gauge for the symmetry under boundary reparametrizations that this action possesses. The gauge that we will fix is slightly different from the one we discussed in the previous section. Here we will start by gauge-fixing θ. A convenient choice is to set $\theta(t,\sigma) = -t + \bar{\theta}(\sigma)$. By arguments similar to those given in the previous section, we again arrive at the gauge-fixed solution (3.10).

The choice of the precise functional form of $\bar{\theta}(\sigma)$ depends on whether $\partial_\sigma\bar{\theta}(\sigma)$ vanishes anywhere or not. Geometrically, the tangent to the boundary at such a point is in the radial direction. Physically, at these points θ is gauge-invariant and the gauge is fixed on ρ, to arrive at the gauge-fixed solution (3.10). The simplest case is that of a boundary which has no such points. More generally, however, there may be several points on the boundary at which the tangent is radially directed. Clearly, the number of such points has to be even because the boundary is a closed curve, unless $\partial_\sigma^2\bar{\theta}(\sigma)$ also vanishes at some point, i.e. it is a point of inflexion. Additionally, there may be points where still higher derivatives of $\bar{\theta}(\sigma)$ also vanish. In the following, we will separately discuss quantization for the two cases: (i) tangent not radially directed at any point on the boundary and (ii) tangent radially directed at one or more points, some of which may also have vanishing higher derivatives of $\bar{\theta}(\sigma)$.

In the following, for simplicity we will restrict our detailed discussion to a single fluid droplet, assuming further that the droplet is centered around the origin in phase plane. At the end of this section, we will comment on extension of this discussion to more general fluid configurations of the type shown in figure 1*b*.

4.1 Boundary profiles with tangent nowhere radially directed

This is the simplest case. A representative example of this class of boundary profiles is shown in figure 1*a*. By a gauge choice, consistent with equation of motion, in this case we may set

$$\theta(t,\sigma) = -t + \bar{\theta}(\sigma)\,, \qquad \bar{\theta}(\sigma) = \sigma\,. \tag{4.1}$$

The canonical equal-time commutation relation (actually the Dirac bracket) for ϕ that follows from the action (3.6) is

$$[\partial_\sigma\phi(t,\sigma), \partial_{\sigma'}\phi(t,\sigma')] = -2i\pi\hbar^2\partial_\sigma\delta(\sigma-\sigma')\,. \tag{4.2}$$

Using (3.5) and (4.1), the commutation relation for ρ follows:

$$[\rho(t,\sigma), \rho(t,\sigma')] = -2i\pi\hbar^2\partial_\sigma\delta(\sigma-\sigma')\,. \tag{4.3}$$

In the above gauge, ρ is time-independent. In terms of modes,

$$\rho(t,\sigma) = \bar{\rho}(\sigma) = \rho_0 + \hbar\sum_{m=1}^{\infty}\left(\alpha_m\, e^{im\sigma} + \alpha_m^\dagger\, e^{-im\sigma}\right). \tag{4.4}$$

The constant term is fixed to be ρ_0 because of the fixed area constraint $\int d\sigma\ \partial_\sigma\theta\rho = 2\pi\rho_0$. The physical degrees of freedom are the complex modes $\alpha_m (m = 1, 2, \ldots)$. Because of (4.3) they satisfy the harmonic oscillator commutation relations

$$[\alpha_m, \alpha_n^\dagger] = m\delta_{mn}\,. \tag{4.5}$$

The hamiltonian is given by

$$H = \frac{1}{4\pi\hbar}\int d\sigma\ \partial_\sigma\theta\ \bar{\rho}^2\,. \tag{4.6}$$

In terms of the modes of $\bar{\rho}$ this reads

$$H = \hbar\sum_{m=1}^{\infty}\alpha_m^\dagger\alpha_m + \frac{\rho_0^2}{2\hbar} + \text{“zero-point energy”}\,. \tag{4.7}$$

The second term in the hamiltonian is the ground state energy. In the fermionic picture, this is the energy of the fermi ground state. The last term is an infinite “zero-point energy”.

The first term in the hamiltonian gives excitation energies. The excited states are constructed from an infinite number of decoupled oscillators with frequencies $1, 2, \ldots$, just like in free string theory. For low energies, these states are in one-to-one correspondence with the spectrum of the fermion theory. However, it is easy to see that this correspondence breaks down at high enough energies, if the number of fermions N is finite, though it may be very large. In fact, the partition function of the fermion theory agrees with that of the collective theory only if we cut-off the oscillator frequency of the collective field[15] at N. This cut-off has to be imposed by hand; it is not a part of the standard quantization of the collective theory. In contrast, as we have argued earlier, in the noncommutative (i.e. exact) formulation of the bosonized theory, the spectrum exactly matches with the fermi theory, for any number of fermions (including one!) without the need for any cut-off.

Another interesting quantity to compute is the Wigner density $u(\rho, \theta, t)$ in some state. It is possible to write down a manifestly gauge-invariant classical expression for $u(\rho, \theta, t)$ in terms of the functions $\rho(\sigma, t)$ and $\theta(\sigma, t)$ which characterize the boundary. We have,

$$u(\rho, \theta, t) = \int d\sigma\ \partial_\sigma\theta(\sigma, t)\,\Theta(\rho(\sigma, t) - \rho)\ \delta(\theta(\sigma, t) - \theta)\,. \tag{4.8}$$

Here Θ is the familiar step-function. One can easily verify from this expression that the equation of motion for u, $(\partial_t - \partial_\theta)u = 0$, implies the equation of motion (3.8) for the boundary. To see that this expression satisfies the constraint $u^2 = u$, we substitute in it

[15] I would like to thank S. Minwalla for pointing this out to me. The calculation of the partition function of N fermions in a harmonic oscillator potential can be equivalently done as the calculation of the partition function of a matrix valued harmonic oscillator, gauged under $\mathrm{U}(N)$. (In fact, the 1/2-BPS sector of $\mathcal{N} = 4$ superYang-Mills is actually a $\mathrm{U}(N)$ one-matrix quantum mechanical system with a harmonic oscillator potential [36, 14].) The latter calculation must take into account only the gauge-invariant states and the partition function over these is given by $Z_N(\beta) = \Pi_{n=1}^N(1 - e^{-\beta n})^{-1}$. Comparing with the well-known partition function of the bosonic system described by the hamiltonian (4.7), namely $Z(\beta) = \Pi_{n=1}^\infty(1 - e^{-\beta n})^{-1}$, we see that agreement requires the cut-off.

JHEP07(2005)064

the gauge-fixed solution (3.10) to the equation of motion. Then, the σ integral in (4.8) can be explicitly done. We get,

$$u(\rho,\theta,t) = \Theta(\bar{\rho}(t+\theta)-\rho)\,, \tag{4.9}$$

which has the desired form.

Upon quantization, the boundary fluctuates, so in the quantum theory a reasonable definition would be in terms of averages in a given state. We define

$$u(\rho,\theta,t) = \langle\Theta(\bar{\rho}(t+\theta)-\rho)\rangle\,. \tag{4.10}$$

The quantum average can be worked out exactly in any state since we are dealing with a free theory. Here we will restrict ourselves to the simplest case of the ground state, $|0>$. In this case $u \equiv u(\rho)$ will turn out to be a function of ρ only. Taking a derivative of (4.10) with respect to ρ, using the fourier representation of the resulting δ-function, and doing the average gives the result

$$\partial_\rho u(\rho) = -\frac{1}{\sqrt{2\pi c}}e^{-(\rho-\rho_0)^2/2c}\,, \tag{4.11}$$

where

$$c = \langle 0|(\bar{\rho}(t+\theta)-\rho_0)^2|0\rangle = \hbar^2\sum_{m=1}^{\infty} m\,. \tag{4.12}$$

To get a finite value for c, one needs to impose a cut-off on the frequency sum by hand. Then, using the boundary condition that $u(\rho)$ vanishes at infinity, we can integrate (4.11) to get

$$u(\rho) = \frac{1}{\sqrt{2\pi c}}\int_\rho^\infty dx\; e^{-(x-\rho_0)^2/2c}\,. \tag{4.13}$$

As a consistency condition, the density obtained above must satisfy the fixed area constraint $\int d\rho\; u(\rho) = \rho_0$. This constraint[16] translates into the condition

$$\frac{\rho_0}{\sqrt{2c}} = \frac{1}{2}\int_{\rho_0/\sqrt{2c}}^\infty d\rho\;(1-\mathrm{erf}(\rho))\,, \tag{4.14}$$

where $\mathrm{erf}(\rho)$ is the error function. It turns out that this condition is always satisfied if $N_0 \le \rho_0/\hbar$, where N_0 is a cut-off (assumed large) on the sum in (4.12). Since $\rho_0/\hbar$ is finite, though it may be large for small $\hbar$, we once again see the need to impose a cut-off on the oscillator frequencies.

The classical density function for the ground state is the step-function $\Theta(\rho_0-\rho)$. Quantum corrections work in the right direction and soften the fall-off to an exponential. However, for finite N the detailed functional form of the density in (4.13) does not match with that of the exact answer for the fermi vacuum given in (2.16). For example, the

[16] One might think that this constraint should be automatically satisfied since, at least formally, one has $\int_0^\infty d\rho\; u = \int_0^\infty d\rho\; <0|\Theta(\bar{\rho}(t+\theta)-\rho)|0> = <0|\bar{\rho}(t+\theta)|0> = \rho_0$. However, it turns out this formal argument does not work since large negative fluctuations can destroy the positivity of $\bar{\rho}(\sigma)$. Hence the need to impose this constraint explicitly.

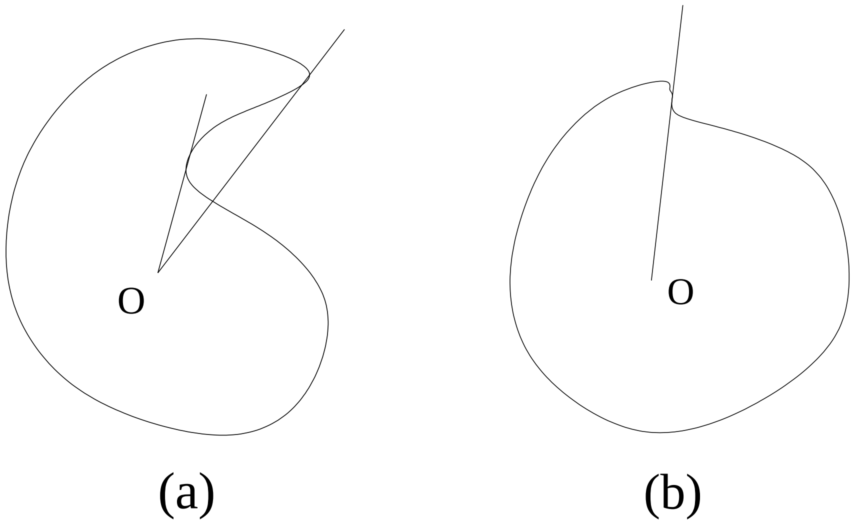

Figure 2: (a) A fluid boundary profile with radially directed tangents at two points. (b) A fluid boundary profile with a radially directed tangent at one point which is also a point of inflexion. Both droplets are centered at the origin O of the phase plane.

integrand in the latter case shows rapid oscillations with a wavelength which decreases as $N^{-1/2}$ for large-N. In contrast, the collective theory answer for the integrand in (4.13) is a simple gaussian which has no such feature. It is important to emphasize here that (4.13) is the exact answer for the density in the ground state in the collective theory. There are no corrections. We conclude that the collective quantum theory calculation does not agree with the exact answer for finite N.

4.2 Boundary profiles with radially directed tangents

Figure 2a shows an example of a boundary profile which has two such points, none of which is a point of inflexion. Figure 2b shows an example of a boundary profile which has just a point of inflexion. As we have discussed, a gauge cannot be completely fixed on θ for such boundaries; gauge-fixing has to be partly done on ρ. As a consequence of this, ρ does not provide a complete description of the gauge-invariant degrees of freedom for such boundaries. In this case, a more convenient set of variables is provided by the gauge-invariant quantities α_m which are defined below:

$$\alpha_m \equiv \frac{1}{2\pi\hbar}\int d\sigma\ \partial_\sigma\phi(\sigma,t)e^{-im(t+\theta(\sigma,t))}\,, \qquad m = 0, \pm 1, \pm 2, \dots\,. \tag{4.15}$$

They are manifestly invariant under time-dependent boundary reparametrizations. Moreover, using (3.5) and the gauge-fixed solution to the equations of motion, (3.10), we get

$$\alpha_m = \frac{1}{2\pi\hbar}\int d\sigma\ \partial_\sigma\bar{\theta}(\sigma)\ \bar{\rho}(\sigma)\ e^{-im\bar{\theta}(\sigma)}\,. \tag{4.16}$$

For $\bar{\theta}(\sigma) = \sigma$, these are identical to the modes used in the previous subsection (with $\hbar\alpha_0 = \rho_0$ and $\alpha_{-m} = \alpha_m^\dagger$) for boundary profiles without any radially directed tangents. For the more general boundary profiles under discussion here, there is a subtlety in inverting (4.16) to express $\bar{\rho}$ in terms of these modes. We have from (4.16)

$$\hbar \sum_m \alpha_m \, e^{im\bar{\theta}(\sigma)} = \int d\sigma' \, \partial_{\sigma'} \bar{\theta}(\sigma') \, \bar{\rho}(\sigma') \, \delta(\bar{\theta}(\sigma') - \bar{\theta}(\sigma)) \,. \tag{4.17}$$

Let us denote the location of zeroes of $\partial_\sigma \bar{\theta}$ by σ_i, where $i = 1, 2, \ldots$. For $\sigma \neq \sigma_i$, we get

$$\hbar \sum_m \alpha_m \, e^{im\bar{\theta}(\sigma)} = \mathrm{sign}(\partial_\sigma \bar{\theta}(\sigma)) \, \bar{\rho}(\sigma), \quad \sigma \neq \sigma_i. \tag{4.18}$$

However, for $\sigma = \sigma_i$ the right hand side vanishes, leading to the constraint

$$\sum_m \alpha_m \, e^{im\bar{\theta}(\sigma_i)} = 0 \,, \tag{4.19}$$

one for each point σ_i. Equations (4.18) and (4.19) express one set of gauge-invariant variables, namely $\{\{\bar{\rho}(\sigma), \, \sigma \neq \sigma_i\}, \{\bar{\theta}(\sigma_i)\}\}$[17] in terms of another, namely $\{\alpha_m\}$. This latter set clearly provides a more convenient starting point for a gauge-invariant description of the quantum dynamics of generic boundary profiles.

As in the previous subsection, quantization begins with the canonical commutation relation (4.2), which is valid for generic boundary profiles. From this we deduce[18] the standard harmonic oscillator commutation relations (4.5) for α_m. Moreover, as before the hamiltonian is given by (4.6). Since $\bar{\rho}(\sigma_i)$ does not contribute to the right hand side, we may use (4.18) in it. This gives precisely the expression (4.7) for the hamiltonian in terms of the modes. We see that the spectrum remains unchanged, and, as before, for finite N it does not agree with the spectrum in the fermionic theory.

Analogous to the computation following (4.8), here also one may wish to compute the Wigner density $u(\rho, \theta, t)$ in some state. First note that (4.8) correctly reproduces the classical density for a generic boundary profile. Using the gauge-fixed solution (3.10) to the equation of motion, we get

$$u(\rho, \theta, t) = \int d\sigma \, \partial_\sigma \bar{\theta}(\sigma) \Theta(\bar{\rho}(\sigma) - \rho) \delta(\bar{\theta}(\sigma) - t - \theta) \,. \tag{4.20}$$

Let $\sigma_k(t + \theta)$, $k = 1, 2, \ldots$ be the points at which the δ-function clicks, i.e. $\bar{\theta}(\sigma_k) = t + \theta$. Then, we get

$$u(\rho, \theta, t) = \sum_{k \neq i} \mathrm{sign}(\partial_{\sigma_k} \bar{\theta}(\sigma_k)) \Theta(\bar{\rho}(\sigma_k) - \rho) \,. \tag{4.21}$$

The sum excludes points $\sigma_k = \sigma_i$ at which $\partial_\sigma \bar{\theta}$ vanishes. It is easy to see that the right hand side precisely equals one in the filled region and is zero otherwise. The exclusion of zeroes of $\partial_\sigma \bar{\theta}$ from the sum as well as the presence of the "sign" function in this formula is crucial to reproduce the correct answer.

[17]The complimentary set, namely $\{\{\bar{\rho}(\sigma_i)\}, \{\bar{\theta}(\sigma), \, \sigma \neq \sigma_i\}\}$, has been gauged away.

[18]One needs to use the relation $\frac{1}{2\pi} \int d\sigma \, \partial_\sigma \bar{\theta}(\sigma) e^{i(m-n)\bar{\theta}(\sigma)} = \delta_{mn}$.

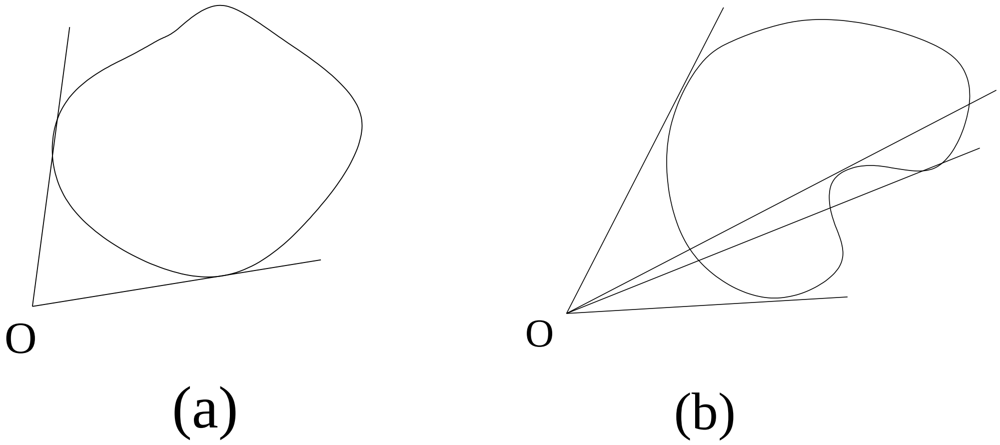

Figure 3: (a) A fluid boundary profile with just the two bounding radially directed tangents. (b) A fluid boundary profile with radially directed tangents other than the two bounding tangents. The origin O of the phase plane is outside both droplets.

In the quantum case, we replace the step-function on the right hand side of (4.21) by the average in some state. For the ground state, the average is independent of σ_k and has the value given in (4.13). The sum is also trivially done since the number of solutions, excluding the points where $\partial_\sigma \bar{\theta}$ vanishes, is always odd with one more plus "sign" than the "minus" sign. The net result for the density is +1 times that in (4.13), i.e. it is identical to that answer. So as there, we conclude that the collective quantum theory calculation does not agree with the exact answer for finite N.

We end this section with the following comment. In the above we have only considered configurations with a single droplet centered around the origin in the phase plane. For more general configurations consisting of several disconnected droplets shown in figure 1b, the density u and hence the action can be written as a sum on the different droplets. The origin of phase plane will be inside at most one of the droplets. Figures 3a and 3b show examples of profiles when the origin of the phase plane is outside the filled region. In such cases there are always at least two radially directed tangents. Furthermore, $\bar{\theta}$ takes a maximum value which is $< 2\pi$. By suitably modifying the above discussion to incorporate these differences, one can easily extend the present analysis to a generic configuration of droplets.

5. Summary and concluding remarks

The noncommutative theory of the Wigner phase space density developed in [29−31] provides an exact bosonization of 2-dimensional non-relativistic fermions. Utilizing the construction of noncommutative solitons of [33], in this paper we have shown that the spectrum of the bosonized theory is identical to the spectrum of the fermion theory. Moreover, given a Wigner density function which solves the equation of motion and constraints of the bosonized theory, we have a precise algorithm for building the occupied levels in the corresponding fermi state. In contrast, we have shown that the collective quantization of fluctuations of fermi fluid droplet boundaries neither reproduces the spectrum nor the details of the phase space density, except perhaps for strictly infinite number of fermions. Although we can explicitly demonstrate this exactly only for the harmonic oscillator potential, we believe the result is valid more generally.

One possible way out of this disagreement with the collective theory could be that the jacobian arising from the complicated change of variables [35] modifies the "classical" action (3.6) at finite N, possibly to all orders in $1/N$ (or equivalently in $\hbar$). A systematic procedure for arriving at such a "corrected" action would be useful, but we have not attempted this here. Even if one is able to find such an action, for finite number of fermions one would still need to impose by hand a cut-off on the collective field oscillator frequencies at N, which is required for the spectrum of the collective theory to match with the spectrum of N fermions. In fact, since it is the oscillators defined in (4.16) that are the appropriate gauge-invariant variables to use for boundary profiles of any shape, it might be useful to reformulate the collective theory directly in terms of N such oscillators, namely the set $\{(\alpha_m,\ \alpha_m^\dagger),\ m = 1, 2, \dots, N\}$. Any "corrected" collective action in terms of this set of variables, which reproduces the results of the fermi theory, at least order-by-order in $\hbar$, is likely to have much of the structure of the exact noncommutative bosonization. In fact, the latter may be a good staring point for investigating this possibility, which we leave for future work.

The present work has implications for probing quantum gravity in the 1/2-BPS sector using supergravity fluctuations [28] and D-branes [23]. As discussed above, exact equivalence to free fermions via AdS/CFT requires the appearance of a noncommutative structure on the gravity side. It would be very interesting if one could relate this noncommutative structure to the cut-off required in quantum gravity.

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